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AN UPPER BOUND FOR THE CARDINALITY OF AN s-DISTANCE SET IN EUCLIDEAN SPACE

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In this paper we show that if X is an s-distance set in $\mathbb{R}^m$ and X is on p concentric spheres then $|X| \le \sum_{i=0}^{2p-1} \binom{m+s-i-1}{s-i}$. Moreover if X is antipodal, then $|X| \le 2\sum_{i=0}^{p-1} \binom{m+s-2i-2}{m-1}$.

1. Introduction

A subset X in a metric space $(M,\, d)$ is called an s-distance set if the cardinality of the set $A=\{d(x,y) \mid x,y\in X,\ x\neq y\}$ is equal to s. In [5] P. Delsarte, J. M. Goethals and J. J. Seidel gave the concept of spherical designs and found the following interesting relations between spherical designs and s-distance sets on spheres. Namely, the cardinality of a spherical $2s$-design X on the sphere S^{m-1} $(\subset \mathbb{R}^m)$ is bounded below by $\binom{m+s-1}{s}+\binom{m+s-2}{s-1}$, and also the cardinality of a spherical $(2s-1)$-design X on the sphere S^{m-1} is bounded below by $2\binom{m+s-2}{s-1}$. On the other hand in the same paper, they showed that the cardinality of an s-distance set X on the sphere S^{m-1} is bounded above by $\binom{m+s-1}{s}+\binom{m+s-2}{s-1}$ and if X is antipodal then the cardinality is bounded above by $2\binom{m+s-2}{s-1}$. They defined a spherical design is tight if the cardinality coincides with one of the lower bounds given above. They proved that a finite set on the sphere S^{m-1} of cardinality $|X|=\binom{m+s-1}{s}+\binom{m+s-2}{s-1}$ is a tight $2s$-design if and only if it is an s-distance set. They also proved that a finite set on the sphere S^{m-1} of cardinality $|X|=2\binom{m+s-2}{s-1}$ is a tight $(2s-1)$-design if and only if it is an antipodal s-distance set.

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The upper bound of the cardinality of an s-distance set in $\mathbb{R}^m$ was studied by Bannai-Bannai-Stanton [2] and Blokhuis [3] independently. They showed that the cardinality of an s-distance set in $\mathbb{R}^m$ is bounded above by $\binom{m+s}{s}$. The concept of Euclidean designs is defined in the paper by P. Delsarte and J. J. Seidel [4]. Delsarte and Seidel proved that the cardinality of a Euclidean $2s$-design on p concentric spheres in $\mathbb{R}^m$ is bounded below by $\sum_{i=0}^{2p-1}\binom{m+s-i-1}{s-i}$ and the cardinality of an antipodal Euclidean $(2s–1)$-design on p concentric spheres in $\mathbb{R}^m$ is bounded below by $2\sum_{i=0}^{p-1}\binom{m+s-2i-2}{m-1}$.

In this paper we prove the following theorem which improves the upper bound for an s-distance set in $\mathbb{R}^m$.

Theorem 1.1.
(1) *Let X be an s-distance set on p concentric spheres in $\mathbb{R}^m$. Then*

$$|X| \le \sum_{i=0}^{2p-1}\binom{m+s-i-1}{s-i}.$$

(2) *Let X be an antipodal s-distance set on p concentric spheres in $\mathbb{R}^m$. Then*

$$|X| \le 2\sum_{i=0}^{p-1}\binom{m+s-2i-2}{m-1}.$$

Remark. If $p = 1$, then $\sum_{i=0}^{2p-1}\binom{m+s-i-1}{s-i} = \binom{m+s-1}{s} + \binom{m+s-2}{s-1}$ and $2\sum_{i=0}^{p-1}\binom{m+s-2i-2}{m-1} = 2\binom{m+s-2}{s-1}$ hold and the bounds coincide with the bounds given by Delsarte, Goethals and Seidel for the spherical case. If $s \le 2p-1$, then $\sum_{i=0}^{2p-1}\binom{m+s-i-1}{s-i} = \sum_{i=0}^{s}\binom{m+s-i-1}{s-i} = \binom{m+s}{s}$. This means that Theorem 1.1 is true for $s \le 2p-1$. Hence if $s=2$ or 3, and $p \ge 2$, then the upper bound given in Theorem 1.1, (1) coincides with the known one, $\binom{m+s}{s}$. If $s \ge 2p$, then $\sum_{i=0}^{2p-1}\binom{m+s-i-1}{s-i} < \binom{m+s}{s}$ and Theorem 1.1, (1) gives a better upper bound.

As for the subsets in $\mathbb{R}^m$ there is an example of a 2-distance set in $\mathbb{R}^8$ whose cardinality is $\binom{8+2}{2}$. This example was found by Lisoněk [7] and it is on 2 concentric spheres. However it is not a tight 4-design as a Euclidean design even though its cardinality coincides with the upper bound.

It is still unknown whether any tight $2s$-design gives an s-distance set or not. This problem seems very important and interesting.

For more information on this subject, see [1] and [4].

In §2 we give basic facts about the vector space of the polynomials on a finite number of concentric spheres in $\mathbb{R}^m$. In §3 we give a proof of Theorem 1.1.

2. Polynomials on p concentric spheres in $\mathbb{R}^m$

First we give notation which will be used in the following and then give basic facts about polynomials on a finite number of concentric spheres (see [4]). Let $S_1, S_2, \dots, S_p$ be spheres in $\mathbb{R}^m$ centered at the origin of $\mathbb{R}^m$ with radii $r_1, r_2, \dots, r_p$ respectively. Let $S = S_1 \cup S_2 \cup \cdots \cup S_p$. Let $P(\mathbb{R}^m)$ be the set of all the polynomials of m variables $x_1, x_2, \dots, x_m$. Let $\mathrm{Hom}_l(\mathbb{R}^m)$ be the set of all the homogeneous polynomials of degree l. We denote the Laplacian $\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \cdots + \frac{\partial^2}{\partial {x_m}^2}$ by Δ. Let $\mathrm{Harm}_l(\mathbb{R}^m)$ be the set of all the harmonic homogeneous polynomials of degree l, i.e., $\mathrm{Harm}_l(\mathbb{R}^m) = \{f \in \mathrm{Hom}_l(\mathbb{R}^m) \mid \Delta f = 0\}$. Let $P(S) = \{f|_S \mid f \in P(\mathbb{R}^m)\}$, $\mathrm{Hom}_l(S) = \{f|_S \mid f \in \mathrm{Hom}_l(\mathbb{R}^m)\}$, $\mathrm{Harm}_l(S) = \{f|_S \mid f \in \mathrm{Harm}_l(\mathbb{R}^m)\}$. For $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ in $\mathbb{R}^m$, the inner product of x and y is denoted by $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$. Let $||x||^2 = \langle x, x\rangle = \sum_{i=1}^m {x_i}^2$.

The following propositions are known.

Proposition 2.1. (See [6])

(i) $Hom_l(\mathbb{R}^m) = Harm_l(\mathbb{R}^m) \oplus ||x||^2 Hom_{l-2}(\mathbb{R}^m)$

(ii) $dim(Hom_l(\mathbb{R}^m)) = \binom{m+l-1}{l} = \binom{m+l-1}{m-1}$

(iii) $dim(Harm_l(\mathbb{R}^m)) = \binom{m+l-1}{l} - \binom{m+l-3}{l-2} = \binom{m+l-1}{m-1} - \binom{m+l-3}{m-1}$

Proposition 2.2. (See [4]) *Let $\rho : P(\mathbb{R}^m) \longrightarrow P(S)$ be the linear map defined by $\rho(f) = f|_S$ for any $f \in P(\mathbb{R}^m)$. Then the following hold.*

(i) *The kernel of ρ is the ideal generated by $\prod_{i=1}^p (||x||^2 - {r_i}^2)$.*

(ii) *$Hom_i(S) \cong Hom_i(\mathbb{R}^m)$, for each non-negative integer i.*

(iii) $\displaystyle\sum_{i=0}^{l} Hom_i(S) = \bigoplus_{i=0}^{2p-1} Hom_{l-i}(S) \cong \bigoplus_{i=0}^{2p-1} Hom_{l-i}(\mathbb{R}^m).$

(iv) $\displaystyle dim\left(\sum_{i=0}^{l} Hom_i(S)\right) = \sum_{i=0}^{2p-1} \binom{m+l-i-1}{l-i}.$

We define some more notations which we will use in this paper. For a vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ whose entries are non-negative integers, we define $|\lambda| = \sum_{i=1}^m \lambda_i$. For any $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, we write $x^\lambda = {x_1}^{\lambda_1} {x_2}^{\lambda_2} \cdots {x_m}^{\lambda_m}$. The next proposition is very elementary but useful.

Proposition 2.3. *Let $u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ be a vector. Then the coefficient of the monomial x^λ in $||x||^{2i} \langle x, u\rangle^{l-2i}$ is equal to*

$$\frac{1}{(\lambda_1)!(\lambda_2)!\cdots(\lambda_m)!}(l-2i)!\ \Delta^i(x^\lambda)|_{x=u}\,,$$

where $\Delta^i(x^\lambda)|_{x=u}$ means that take i times the Laplacian of x^λ and substitute $x = u$.

The following lemma, which may be well known, is useful. We use some modifications of this Lemma in our proof of Theorem 1.1.

Lemma 2.4. *Let* $u=(u_1,u_2,\ldots,u_m)$ *be a vector in* $\mathbb{R}^m$. *Assume that there exist real numbers* $c_1,c_2,\ldots,c_{[\frac{l}{2}]}$ *satisfying the following equation*

$$\langle x,u\rangle^l = \sum_{i=1}^{[\frac{l}{2}]} c_i||x||^{2i}\langle x,u\rangle^{l-2i}$$

for any $x=(x_1,x_2,\ldots,x_m)\in\mathbb{R}^m$. *Then* $\varphi(u)=0$ *for any* $\varphi\in \mathrm{Harm}_l(\mathbb{R}^m)$

Proof. The coefficient of the monomial ${x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_m}^{\lambda_m}$ in $\langle x,u\rangle^l$ is equal to

$$\frac{l!}{(\lambda_1)!(\lambda_2)!\cdots(\lambda_m)!}\,{u_1}^{\lambda_1}{u_2}^{\lambda_2}\cdots{u_m}^{\lambda_m},$$

for any non-negative integers $\lambda_1,\lambda_2,\ldots,\lambda_m$ satisfying $\lambda_1+\lambda_2+\cdots+\lambda_m=l$. Hence Proposition 2.3 implies the following equation

$${u_1}^{\lambda_1}{u_2}^{\lambda_2}\cdots{u_m}^{\lambda_m} = \frac{1}{l!}\sum_{i=1}^{[\frac{l}{2}]} c_i(l-2i)!\ \Delta^i({x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_m}^{\lambda_m})|_{x=u}\ .$$

Since Δ^i is a linear operator we have

$$f(u) = \frac{1}{l!}\sum_{i=1}^{[\frac{l}{2}]} c_i(l-2i)!\ (\Delta^i\ f)(u),$$

for any homogeneous polynomial $f\in \mathrm{Hom}_l(\mathbb{R}^m)$. In particular if φ is a harmonic polynomial in $\mathrm{Harm}_l(\mathbb{R}^m)$ we have $\varphi(u)=0$. ∎

3. Proof of Theorem 1.1

First we prove Theorem 1.1, (1). Let X be an s-distance set in $\mathbb{R}^m$. Let $A=A(X)=\{d(u,v)\mid u,v\in X,\ u\neq v\}$, where $d(u,v)=\sqrt{\langle u-v,u-v\rangle}=||u-v||$. Then by the assumption on X we have $|A|=s$. Let $A=\{\alpha_1,\alpha_2,\ldots,\alpha_s\}$. For each $u\in X$, we define a polynomial $F_u\in P(\mathbb{R}^m)$ by

$$F_u(x) = \prod_{i=1}^{s}(||x-u||^2-{\alpha_i}^2).$$

Then we have

$$F_u(v) = \delta_{u,v}(-1)^s \prod_{i=1}^{m} {\alpha_i}^2 \tag{3.1}$$

for any u, $v \in X$. By (3.1) the set of polynomials $\mathcal{F}_X = \{F_u \mid u \in X\}$ is linearly independent in $P(\mathbb{R}^m)$. For each $u \in X$, the polynomial F_u is a polynomial of highest degree $2s$, that is, $F_u \in \sum_{i=0}^{2s} \mathrm{Hom}\,_i(\mathbb{R}^m)$. Since $\mathcal{F}_X$ is a set of linearly independent polynomials in a finite dimensional vector space $\sum_{i=0}^{2s} \mathrm{Hom}\,_i(\mathbb{R}^m)$, X has to be a finite set. Let $R = R_X = \{||u|| \mid u \in X\}$. Then R consists of a finite number of real numbers. Without loss of generality, we may assume that $0 \notin R$. Let $|R| = p$ and $R = \{r_1, r_2, \ldots, r_p\}$. For each i with $1 \le i \le p$, let S_i be the sphere in $\mathbb{R}^m$ with center at the origin with radius r_i. Let $S = S_1 \cup S_2 \cup \cdots \cup S_p$. Then (3.1) also implies that $\mathcal{F}_X$ is linearly independent as polynomials in $P(S)$. Let $\mathcal{F}_X(S) = \{F_u|_S \mid u \in X\}$. Then $|X| = \dim(\langle \mathcal{F}_X(S)\rangle)$. In the following we look for the upper bounds for $\dim(\langle \mathcal{F}_X(S)\rangle)$. As mentioned in the Remark right after Theorem 1.1, if $s \le 2p-1$ then Theorem 1.1 is true. From now on we assume $s \ge 2p$.

Lemma 3.1.

(i) $\langle \mathcal{F}_X(S)\rangle \subset \bigoplus_{i=0}^{2p-1} Hom_{s-i}(S) + \sum_{i=1}^{p-1} ||x||^{2i} Hom_{s-i}(S)$.

(ii) $$\bigoplus_{i=0}^{2p-1} Hom_{s-i}(S) + \sum_{i=1}^{p-1} ||x||^{2i} Hom_{s-i}(S) = \bigoplus_{i=0}^{2p-1} Hom_{s-i}(S) + \sum_{i=1}^{p-1} \sum_{j=0}^{p-1-i} ||x||^{2(i+j)} Harm_{s-i-2j}(S)$$

Proof. We have the following expression for the polynomial F_u:

$$F_u(x) = \sum_{i=0}^{s} \beta_{s-i}^{(u)} (||x||^2 - 2\langle x, u\rangle)^i,$$

where $\beta_i^{(u)}, 0 \le i \le s$ is the elementary symmetric polynomial of $\{||u||^2 - \alpha_1^2, \ldots, ||u||^2 - \alpha_s^2\}$ of degree i. In particular $\beta_0^{(u)} = 1$. Therefore,

$$\mathcal{F}_X \subset \langle \{||x||^{2i} \langle x, u\rangle^j \mid i + j \le s\}\rangle \subseteq \sum_{\substack{i+j \le s \\ 0 \le i,j}} ||x||^{2i} \mathrm{Hom}_j(\mathbb{R}^m).$$

If $i \neq 0$, $i+j < s$, and $2i+j \ge s+1$, then we have $i > 2i+j-s \ge 1$ and $s-i-j > 0$. Therefore we have

$$||x||^{2i} \mathrm{Hom}_j(\mathbb{R}^m) = ||x||^{2(2i+j-s)} ||x||^{2(s-i-j)} \mathrm{Hom}_j(\mathbb{R}^m) \subset ||x||^{2(2i+j-s)} \mathrm{Hom}_{s-(2i+j-s)}(\mathbb{R}^m).$$

Hence we have

$$\langle \mathcal{F}_X \rangle \subset \bigoplus_{i=0}^{s} \mathrm{Hom}_i(\mathbb{R}^m) + \sum_{i=0}^{s} ||x||^{2i} \mathrm{Hom}_{s-i}(\mathbb{R}^m).$$

Hence by Proposition 2.2, we have

$$\langle \mathcal{F}_X(S) \rangle \subset \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=0}^{s} ||x||^{2i} \mathrm{Hom}_{s-i}(S). \tag{3.2}$$

Next, we will show that

$$||x||^{2j} \mathrm{Hom}_{s-j}(S) \subset \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=0}^{j-1} ||x||^{2i} \mathrm{Hom}_{s-i}(S), \tag{3.3}$$

for any $j \geq p$. By Proposition 2.2, $\prod_{l=1}^{p}(||x||^2 - r_l^2)$ generates the kernel of the linear map ρ defined by the restriction of the polynomials on $\mathbb{R}^m$ to S. Hence, as a polynomial on S, $||x||^{2p}$ is a linear combination of $||x||^{2i}$, $i=0,1,2,\ldots,p-1$. Therefore we have

$$||x||^{2p} \mathrm{Hom}_l(S) \subset \sum_{i=0}^{p-1} ||x||^{2i} \mathrm{Hom}_l(S). \tag{3.4}$$

for any integer $l \geq 0$. Now we assume $j \geq p$. Then by (3.4) we have

$$\begin{aligned} ||x||^{2j} \mathrm{Hom}_{s-j}(S) &= ||x||^{2p} ||x||^{2(j-p)} \mathrm{Hom}_{s-j}(S) \subset \\ \sum_{i=0}^{p-1} ||x||^{2i} ||x||^{2(j-p)} \mathrm{Hom}_{s-j}(S) &= \sum_{k=1}^{p} ||x||^{2(j-k)} \mathrm{Hom}_{s-j}(S). \end{aligned}$$

If $j \leq 2k$, then

$$||x||^{2(j-k)} \mathrm{Hom}_{s-j}(S) \subset \mathrm{Hom}_{s-(2k-j)}(S) \subset \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S). \tag{3.5}$$

If $j \geq 2k+1$, then

$$\begin{aligned} ||x||^{2(j-k)} \mathrm{Hom}_{s-j}(S) &= ||x||^{2(j-2k)} ||x||^{2k} \mathrm{Hom}_{s-j}(S) \subset \\ &||x||^{2(j-2k)} \mathrm{Hom}_{s-(j-2k)}(S) \subset \sum_{i=1}^{j-1} ||x||^{2i} \mathrm{Hom}_{s-i}(S) \end{aligned} \tag{3.6}$$

because $k \geq 1$. (3.5) and (3.6) imply (3.3). Induction on j using (3.3) implies

$$\sum_{i=0}^{s} ||x||^{2i}\mathrm{Hom}_{s-i}(S) \subset \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=1}^{p-1} ||x||^{2i}\mathrm{Hom}_{s-i}(S). \tag{3.7}$$

Equations (3.2) and (3.7) imply Lemma 3.1, (i).
Next we prove Lemma 3.1, (ii). It is enough to show that

$$\bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=1}^{p-1} ||x||^{2i}\mathrm{Hom}_{s-i}(S) \subseteq \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=1}^{p-1} \sum_{j=0}^{p-1-i} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S).$$

Proposition 2.1, (i) implies

$$||x||^{2i}\mathrm{Hom}_{s-i}(S) = \sum_{j=0}^{p-i-1} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S) + ||x||^{2p}\mathrm{Hom}_{s+i-2p}(S), \tag{3.8}$$

for any i with $1 \leq i \leq p-1$ (we note that $s \geq 2p$). Then by (3.4) we have

$$||x||^{2p}\mathrm{Hom}_{s+i-2p}(S) \subset \sum_{l=0}^{p-1} ||x||^{2l}\mathrm{Hom}_{s+i-2p}(S). \tag{3.9}$$

(3.9) and (3.9) imply

$$||x||^{2i}\mathrm{Hom}_{s-i}(S) \subset \sum_{j=0}^{p-i-1} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S) + \sum_{l=0}^{p-1} ||x||^{2l}\mathrm{Hom}_{s+i-2p}(S), \tag{3.10}$$

for any i with $1 \leq i \leq p-1$. Next we will show

$$||x||^{2l}\mathrm{Hom}_{s+i-2p}(S) \subset \bigoplus_{k=0}^{2p-1} \mathrm{Hom}_{s-k}(S) + \sum_{k=1}^{i-1} ||x||^{2k}\mathrm{Hom}_{s-k}(S) \tag{3.11}$$

for any i, l with $1 \leq i \leq p-1$ and $0 \leq l \leq p-1$.
If $i + 2l \leq 2p$, then

$$||x||^{2l}\mathrm{Hom}_{s+i-2p}(S) \subset \mathrm{Hom}_{s-(2p-2l-i)}(S) \subset \bigoplus_{k=0}^{2p-1} \mathrm{Hom}_{s-k}(S). \tag{3.12}$$

If $i+2l \geq 2p+1$, then $i+2l-2p \geq 1$. On the other hand $2p-i-l \geq 2$. Hence we have

$$||x||^{2l}\mathrm{Hom}_{s+i-2p}(S) = ||x||^{2(i+2l-2p)}||x||^{2(2p-i-l)}\mathrm{Hom}_{s+i-2p}(S)$$

$$\subset ||x||^{2(i+2l-2p)}\mathrm{Hom}_{s-(i+2l-2p)}(S) \subset \sum_{k=1}^{i-1} ||x||^{2k}\mathrm{Hom}_{s-k}(S), \tag{3.13}$$

because $l-p<0$. Then (3.12) and (3.13) imply (3.11). Then (3.11) and (3.11) imply

$$||x||^{2i}\mathrm{Hom}_{s-i}(S) \subset \bigoplus_{k=0}^{2p-1} \mathrm{Hom}_{s-k}(S) + \sum_{j=0}^{p-1-i} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S)$$

$$+\sum_{k=1}^{i-1} ||x||^{2k}\mathrm{Hom}_{s-k}(S). \tag{3.14}$$

Then induction on i using (3.14) implies Lemma 3.1, (ii). ∎

Now we will start to prove Theorem 1.1, (1). Our proof will be devided into two cases.

Case 1: We assume that $\langle \mathcal{F}_X(S)\rangle \cap \bigoplus_{i=1}^{p-1}\mathrm{Hom}_{s-i}(S)=\{0\}$ holds. Then Proposition 2.2 and Lemma 3.1 imply

$$\langle \mathcal{F}_X(S)\rangle \oplus \bigoplus_{i=1}^{p-1}\mathrm{Hom}_{s-i}(S) \subset$$

$$\bigoplus_{i=0}^{2p-1}\mathrm{Hom}_{s-i}(S) + \sum_{i=1}^{p-1}\sum_{j=0}^{p-1-i} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S).$$

Hence by Proposition 2.1 and Proposition 2.2 we have

$$\begin{aligned}
|X| &= \dim(\mathcal{F}_X(S)) \\
&\leq \sum_{i=p}^{2p-1}\dim(\mathrm{Hom}_{s-i}(\mathbb{R}^m)) + \sum_{i=1}^{p-1}\sum_{j=0}^{p-1-i}\dim(\mathrm{Harm}_{s-i-2j}(\mathbb{R}^m)) \\
&= \sum_{i=p}^{2p-1}\binom{m+s-i-1}{m-1} + \sum_{i=1}^{p-1}\binom{m+s-i-1}{m-1} - \sum_{i=p+1}^{2p-1}\binom{m+s-i-1}{m-1} \\
&= \sum_{i=1}^{p}\binom{m+s-i-1}{m-1} < \sum_{i=0}^{2p-1}\binom{m+s-i-1}{m-1}.
\end{aligned}$$

Hence in this case we have Theorem 1.1, (1).

Case 2: We assume that $\langle \mathcal{F}_X(S)\rangle \cap \bigoplus_{i=1}^{p-1} \mathrm{Hom}_{s-i}(S) \supsetneq \{0\}$ holds. In this case our goal is to prove the following Lemma.

Lemma 3.2. *If*

$$\langle \mathcal{F}_X(S)\rangle \cap \bigoplus_{i=1}^{p-1} Hom_{s-i}(S) \supsetneq \{0\}$$

holds, then there exists a subspace $W \subset \sum_{i=1}^{p-1} Hom_{s-i}(S)$ *satisfying the following conditions:*

(i) $\langle \mathcal{F}_X(S)\rangle \cap W = \{0\}$,

(ii) $\dim W = \dim \left(\sum_{i=1}^{p-1} \sum_{j=0}^{p-i-1} ||x||^{2(i+j)} \mathrm{Harm}_{s-i-2j}(S) \right)$.

If we prove Lemma 3.2 then Theorem 1.1, (1) is obtained as follows. Proposition 2.2, Lemma 3.1 and Lemma 3.2 imply

$$\langle \mathcal{F}_X(S)\rangle \oplus W \subset \bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S) + \sum_{i=1}^{p-1} \sum_{j=0}^{p-1-i} ||x||^{2(i+j)} \mathrm{Harm}_{s-i-2j}(S).$$

Then we have

$$\begin{aligned}
\dim(\langle \mathcal{F}_X(S)\rangle) + \dim W &= \dim\left(\langle \mathcal{F}_X(S)\rangle + W\right) \\
&\leq \dim\left(\bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S)\right) + \dim\left(\sum_{i=1}^{p-1} \sum_{j=0}^{p-1-i} ||x||^{2(i+j)} \mathrm{Harm}_{s-i-2j}(S)\right) \\
&= \dim\left(\bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S)\right) + \dim W.
\end{aligned}$$

Hence we have

$$|X| = \dim(\langle \mathcal{F}_X(S)\rangle) \leq \dim\left(\bigoplus_{i=0}^{2p-1} \mathrm{Hom}_{s-i}(S)\right).$$

Then Proposition 2.2 implies Theorem 1.1, (1).

In the following, we construct a subspace W which satisfies the conditions in Lemma 3.2.

By the assumption there exist a nonzero polynomial $g(x) \in \langle \mathcal{F}_X(S) \rangle \cap \bigoplus_{i=1}^{p-1} \mathrm{Hom}_{s-i}(S)$. Then we can assume $g(x) \in \bigoplus_{i=1}^{p-1} \mathrm{Hom}_{s-i}(\mathbb{R}^m)$ and

$$\sum_{u \in X} a_u F_u(x) = g(x) + f(x) \prod_{i=1}^{p} (||x||^2 - {r_i}^2) \tag{3.15}$$

for any $x \in \mathbb{R}^m$ with some real numbers a_u, $u \in X$ and a polynomial $f(x) \in P(\mathbb{R}^m)$ whose leading term is of degree $2(s-p)$. Let $f(x) = \sum_{i=0}^{2(s-p)} \sum_{|\lambda|=i} b_\lambda x^\lambda$. Let us express

$$\sum_{u \in X} a_u F_u(x) = \sum_{u \in X} a_u \sum_{i=0}^{s} \beta_{s-i}^{(u)} (||x||^2 - 2\langle x, u \rangle)^i,$$

where $\beta_i^{(u)}$ is the elementary symmetric polynomial of degree i for $||u||^2 - \alpha_1^2, ||u||^2 - \alpha_2^2, \ldots, ||u||^2 - \alpha_s^2$. In particular $\beta_0^{(u)} = 1$. We also express

$$\prod_{j=1}^{p} (||x||^2 - {r_j}^2) = \sum_{j=0}^{p} \delta_{p-j} ||x||^{2j},$$

where δ_i is the elementary symmetric polynomial of degree i for $-r_1^2, -r_2^2, \ldots, -r_p^2$. In particular $\delta_0 = 1$. With the notation given above we have

$$\sum_{u \in X} a_u F_u(x) = \sum_{u \in X} a_u \sum_{l=0}^{2s} \sum_{j=\max\{0, l-s\}}^{[\frac{l}{2}]} (-2)^{l-2j} \binom{l-j}{j} \beta_{j-l+s}^{(u)} ||x||^{2j} \langle x, u \rangle^{l-2j} \tag{3.16}$$

and

$$f(x) \prod_{j=1}^{p} (||x||^2 - {r_j}^2) = \left(\sum_{i=0}^{2s-2p} \sum_{|\lambda|=i} b_\lambda x^\lambda \right) \left(\sum_{j=0}^{p} \delta_{p-j} ||x||^{2j} \right)$$

$$= \sum_{l=0}^{2s} \sum_{\substack{0 \le j \le p \\ l-2(s-p) \le 2j \le l}} \sum_{|\lambda| = l-2j} \delta_{p-j} b_\lambda ||x||^{2j} x^\lambda \tag{3.17}$$

Since $g(x)\in\bigoplus_{i=1}^{p-1}\mathrm{Hom}_{s-i}(\mathbb{R}^m)$ we can prove the following (using equations (3.15), (3.16) and (3.17)):

$$\sum_{j=0}^{\min\{p,[\frac{l}{2}]\}} \delta_{p-j}||x||^{2j} \sum_{|\lambda|=l-2j} b_\lambda x^\lambda = (-2)^l \sum_{j=0}^{[\frac{l}{2}]} 2^{-2j}\binom{l-j}{j} \sum_{u\in X} a_u \beta^{(u)}_{s-l+j}||x||^{2j}\langle x,u\rangle^{l-2j}, \quad 0\le l\le s-p, \tag{3.18}$$

$$\sum_{0\le j\le \min\{p,\frac{2s-l}{2}\}} \delta_j ||x||^{2(p-j)} \sum_{|\lambda|=l-2p+2j} b_\lambda x^\lambda = (-2)^{2s-l}||x||^{2(l-s)} \sum_{j=0}^{[\frac{2s-l}{2}]} 2^{-2j}\binom{s-j}{j+l-s}\sum_{u\in X} a_u\beta^{(u)}_j ||x||^{2j}\langle x,u\rangle^{2s-l-2j}, \quad s\le l\le 2s. \tag{3.19}$$

Let $l=s+i$, $0\le i\le s$, in (3.19). Then we have

$$\sum_{0\le j\le \min\{p,\frac{s-i}{2}\}} \delta_j ||x||^{2(p-j)} \sum_{|\lambda|=s-2p+i+2j} b_\lambda x^\lambda = (-2)^{s-i}||x||^{2i} \sum_{j=0}^{[\frac{s-i}{2}]} 2^{-2j}\binom{s-j}{i+j}\sum_{u\in X} a_u\beta^{(u)}_j ||x||^{2j}\langle x,u\rangle^{s-i-2j}. \tag{3.20}$$

We have the following proposition.

Proposition 3.3. *The assumption and notation are as given before. Then the following conditions hold:*
(i) *For any i with $0\le i\le s-p$ the polynomial $\sum_{|\lambda|=i} b_\lambda x^\lambda$ is a linear combination of the polynomials $||x||^{2j}\langle x,u\rangle^{i-2j}$, $0\le j\le [\frac{i}{2}]$, $u\in X$.*
(ii) *For any $s-p\le i\le 2(s-p)$*

$$\sum_{|\lambda|=i} b_\lambda x^\lambda = ||x||^{2(i-(s-p))}\sum_{u\in X} a_u B_{u,2(s-p)-i}(x),$$

where $B_{u,k}(x)$ is defined for $k=0,\dots,s-p$ and is a homogeneous polynomial of degree k having the following expression:

$$B_{u,k}(x) = \sum_{0\le j\le [\frac{k}{2}]} C_{k,j}(||u||^2)||x||^{2j}\langle x,u\rangle^{k-2j},$$

where $C_{k,j}(||u||^2)$ is a linear combination of $\beta_l^{(u)}$, $0\le l\le j$ whose coefficients only depend on s, p, k, l, j and the radii $r_1,\dots,r_p$ of the given concentric spheres. In particular $C_{k,0}(||u||^2)=(-2)^k\binom{s}{k}$, which depends only on s and k.

Proof. (i) We use (3.18). Put $l=0$ in (3.18), then we have $\sum_{|\lambda|=0} b_\lambda x^\lambda = \delta_p^{-1}\sum_{u\in X} a_u\beta_s^{(u)}$. Hence (i) is true for $i=0$. We can prove (i) by induction on i using (3.18) with $l=i$.
(ii) We use (3.19). Put $l=2s$ in (3.19). Then we have

$$\sum_{|\lambda|=2(s-p)} b_\lambda x^\lambda = ||x||^{2(s-p)}\sum_{u\in X} a_u.$$

Hence $B_{u,0}(x)\equiv 1$. Put $l=2s-1$ in (3.19). Then we have

$$\sum_{|\lambda|=2(s-p)-1} b_\lambda x^\lambda = -2||x||^{2(s-p-1)}\binom{s}{s-1}\sum_{u\in X} a_u\langle x,u\rangle.$$

Hence $B_{u,1}(x)=-2\binom{s}{s-1}\langle x,u\rangle$. For any k with $0\le k\le s-p$, put $l=2s-k$ in (3.19), then we have

$$\begin{aligned}||x||^{2p}&\sum_{|\lambda|=2(s-p)-k} b_\lambda x^\lambda = \\ &(-2)^k||x||^{2(s-k)}\sum_{j=0}^{[\frac{k}{2}]} 2^{-2j}\binom{s-j}{j+s-k}\sum_{u\in X} a_u\beta_j^{(u)}||x||^{2j}\langle x,u\rangle^{k-2j} \\ &\qquad - \sum_{j=1}^{\min\{p,\ [\frac{k}{2}]\}} \delta_j||x||^{2(p-j)}\sum_{|\lambda|=2(s-p)-k+2j} b_\lambda x^\lambda.\end{aligned}$$

Then we can prove (ii) by induction on $k=2(s-p)-i$. ∎

For each k with $0\le k\le s-p-1$, let us define a polynomial $B_k(x)$ of degree k by

$$B_k(x) = \sum_{|\lambda|=k} b_\lambda x^\lambda.$$

We express (3.20) with $0 \le i \le p-1$ using $B_k(x)$ and $B_{u,k}(x)$. Then we get

$$\sum_{j=0}^{[\frac{p-i-1}{2}]} \delta_j ||x||^{2(p-j)} B_{s-2p+2j+i}(x) + \sum_{j=[\frac{p-i+1}{2}]}^{\min\{p,[\frac{s-i}{2}]\}} \delta_j ||x||^{2(i+j)} \sum_{u\in X} a_u B_{u,s-i-2j}(x) = (-2)^{s-i} ||x||^{2i} \sum_{j=0}^{[\frac{s-i}{2}]} 2^{-2j} \binom{s-j}{i+j} \sum_{u\in X} a_u \beta_j^{(u)} ||x||^{2j} \langle x,u\rangle^{s-i-2j}, \tag{3.21}$$

for $0 \le i \le p-1$. Then by Proposition 3.3 we have

$$\sum_{j=0}^{[\frac{p-i-1}{2}]} \delta_j ||x||^{2(p-i-j)} B_{s-2p+2j+i}(x) = (-2)^{s-i} \sum_{j=0}^{[\frac{s-i}{2}]} 2^{-2j} \binom{s-j}{i+j} \sum_{u\in X} a_u \beta_j^{(u)} ||x||^{2j} \langle x,u\rangle^{s-i-2j} - \sum_{j=[\frac{p-i+1}{2}]}^{\min\{p,[\frac{s-i}{2}]\}} \delta_j ||x||^{2j} \sum_{u\in X} a_u \sum_{k=0}^{[\frac{s-i-2j}{2}]} C_{s-i-2j,k}(||u||^2) ||x||^{2k} \langle x,u\rangle^{s-i-2j-2k}. \tag{3.22}$$

Hence for any i with $1 \le i \le p-1$ the coefficient of the term $||x||^{2j}\langle x,u\rangle^{s-i-2j}$ in the right hand side of the equation (3.22) is given by

$$(-2)^{s-i-2j} \binom{s-j}{i+j} a_u \beta_j^{(u)} + \text{a linear combination of } \{\beta_l^{(u)},\ 0 \le l \le j-1\}.$$

Then we can express the right hand side of the equation (3.22) in the following way.

$$\sum_{j=0}^{[\frac{2p-i-3}{2}]} \sum_{u\in X} a_u g_{i,j}(||u||^2) ||x||^{2j} \langle x,u\rangle^{s-i-2j} + \text{ a linear combination of the terms } ||x||^{2j}\langle x,u\rangle^{s-i-2j} \text{ with } j \ge \left[\frac{2p-i-1}{2}\right], \tag{3.23}$$

where $g_{i,j}(||u||^2)$ is a linear combination of $\beta_l^{(u)}$, $0 \le l \le j$. More precisely

$$g_{i,j}(||u||^2) = (-2)^{s-i-2j}\binom{s-j}{i+j}\beta_j^{(u)} + \text{a linear combination of } \{\beta_l^{(u)},\ 0 \le l \le j-1\}.$$

By definition, $\beta_l^{(u)}$ is the elementary symmetric polynomial of $\{||u||^2 - \alpha_1^2, \dots, ||u||^2 - \alpha_s^2\}$ of degree l. Hence $g_{i,j}(t)$ is a polynomial in one variable t of degree j with the following form:

$$g_{i,j}(t) = (-2)^{s-i-2j}\binom{s-j}{i+j}t^j + \text{ terms with } t^l,\ l \le j-1.$$

Note that the polynomial $g_{i,j}(t)$ of degree j defined above depends only on $\alpha_1, \dots, \alpha_s$ and $r_1, \dots, r_p$ and $i,\ j$.

Let $U_{s-i}^{(\le k)}$ be a subspace of $\mathrm{Hom}_{s-i}(\mathbb{R}^m)$ defined by

$$U_{s-i}^{(\le k)} = \left\langle ||x||^{2j}\langle x, u\rangle^{s-i-2j} \;\middle|\; 0 \le s-i-2j \le k \right\rangle.$$

Let us define a polynomial in $\mathrm{Hom}_{s-i}(\mathbb{R}^m)$ by

$$\varPhi_{s-i}(x) = \sum_{l=0}^{[\frac{2p-i-3}{2}]}\sum_{u\in X} a_u g_{i,l}(||u||^2)||x||^{2l}\langle x, u\rangle^{s-i-2l}, \quad 0 \le i \le p.$$

Then (3.22) and (3.23) imply

$$\varPhi_{s-i}(x) - \sum_{l=0}^{[\frac{p-i-1}{2}]} \delta_l ||x||^{2(p-i-l)} B_{s-2p+2l+i}(x) \in U_{s-i}^{(\le s-2p+2)}. \tag{3.24}$$

We have the following lemma.

Lemma 3.4. *Assumption and notation are as given before. For each i and j with $1 \le i \le p-1$, $0 \le j \le p-i-1$ there exists a polynomial $G_{i,j}(t)$ of degree j in one variable t satisfying the following condition*

$$\sum_{u\in X} a_u G_{i,j}(||u||^2)\langle x, u\rangle^{s-i-2j} \in U_{s-i-2j}^{(\le s-i-2j-2)}.$$

Moreover the polynomial $G_{i,j}(t)$ does not depend on the choice of the polynomial $g(x) \in \langle \mathcal{F}_X(S)\rangle \cap \sum_{i=1}^{p-1} Hom_{s-i}(S)$.

Proof. Let $i=p-1$ in (3.24). Then we have

$$\Phi_{s-p+1}(x) - ||x||^2 B_{s-p-1}(x) \in U_{s-p+1}^{(\leq s-2p+2)}. \tag{3.25}$$

Let $i=p-2$ in (3.24). Then we have

$$\Phi_{s-p+2}(x) - ||x||^4 B_{s-p-2}(x) \in U_{s-p+2}^{(\leq s-2p+2)}.$$

Let $i=p-3$ in (3.24). Then we have

$$\Phi_{s-p+3}(x) - ||x||^6 B_{s-p-3}(x) - \delta_1 ||x||^4 B_{s-p-1}(x) \in U_{s-p+3}^{(\leq s-2p+2)}.$$

Then (3.25) implies

$$\Phi_{s-p+3}(x) - \delta_1 ||x||^2 \Phi_{s-p+1}(x) - ||x||^6 B_{s-p-3}(x) \in U_{s-p+3}^{(\leq s-2p+2)}.$$

Thus induction on l shows that for any l with $1 \leq l \leq p-1$ the following holds:

$$\Phi_{s-p+l}(x) + \sum_{j=1}^{[\frac{l-1}{2}]} d_{l,j} ||x||^{2j} \Phi_{s-p+l-2j}(x) - ||x||^{2l} B_{s-p-l}(x) \in U_{s-p+l}^{(\leq s-2p+2)},$$

with some constants $d_{l,j}$, $1 \leq j \leq [\frac{l-1}{2}]$ which depend only on $\delta_l, \ldots, \delta_p$. Hence we have

$$\Phi_{s-i}(x) + \sum_{j=1}^{[\frac{p-i-1}{2}]} d_{p-i,j} ||x||^{2j} \Phi_{s-i-2j}(x) \equiv 0 \mod U_{s-i}^{(\leq s-2p+i)}$$

for $i=1,\ldots,p-1$. Then we have

$$\Delta^l \left(\Phi_{s-i}(x) + \sum_{j=1}^{[\frac{p-i-1}{2}]} d_{p-i,j} ||x||^{2j} \Phi_{s-i-2j}(x) \right) \equiv 0 \mod U_{s-i-2l}^{(\leq s-2p+i)}, \tag{3.26}$$

for any l with $0 \leq l \leq p-i-1$. On the other hand compute

$$\Phi_{s-i}(x) + \sum_{j=1}^{[\frac{p-i-1}{2}]} d_{p-i,j} ||x||^{2j} \Phi_{s-i-2j}(x)$$

using the definition of $\Phi_{s-i}(x)$. Then we can show that $\Phi_{s-i}(x) + \sum_{j=1}^{[\frac{p-i-1}{2}]} d_{p-i,j} ||x||^{2j} \Phi_{s-i-2j}(x)$ has the following form:

$$\sum_{k=0}^{p-i-1} \sum_{u \in X} a_u h_{i,k}(||u||^2) ||x||^{2k} \langle x, u \rangle^{s-i-2k},$$

where $h_{i,k}(t)$ is a polynomial of degree k with the following form:

$$h_{i,k}(||u||^2) = (-2)^{s-i-2k}\binom{s-k}{i+k}||u||^{2k} + \text{ terms with } ||u||^{2j},\ 0 \le j \le k-1. \tag{3.27}$$

In general the following holds:

$$\Delta(||x||^{2l}\langle x,u\rangle^k) = 2l(m+2l+2k-2)||x||^{2(l-1)}\langle x,u\rangle^k + k(k-1)||u||^2||x||^{2l}\langle x,u\rangle^{k-2}.$$

Therefore, for any l with $0 \le l \le p-i-1$ we have

$$\Delta^l\left(\Phi_{s-i}(x) + \sum_{j=1}^{[\frac{p-i-1}{2}]} d_{p-i,j}||x||^{2j}\Phi_{s-i-2j}(x)\right) \equiv \sum_{u\in X} a_u G_{i,l}(||u||^2)\langle x,u\rangle^{s-i-2l} \quad (\mathrm{mod}\ U_{s-i-2l}^{\le s-i-2l-2}), \tag{3.28}$$

where each $G_{i,l}(t)$ is a polynomial in one variable t of degree l which is independent of a_u, $u\in X$. Then, (3.26) and (3.28) imply Lemma 3.4. ∎

Let W be a subspace in $\sum_{i=1}^{p-1}\mathrm{Hom}_{s-i}(S)$ defined by

$$W = \sum_{i=1}^{p-1}\sum_{j=0}^{p-i-1} G_{i,j}(||x||^2)\mathrm{Harm}_{s-i-2j}(S),$$

where $G_{i,j}(t)$, $1\le i\le p-1$, $0\le j\le p-i-1$, are the polynomials of degree j given in (3.28).

We will show that the subspace W defined above satisfies the conditions in Lemma 3.2.

Proof of Lemma 3.2.
Let $g(x)\in\langle\mathcal{F}_X(S)\rangle\cap W$. Then we may assume

$$g(x) \in \left(\sum_{i=1}^{p-1}\sum_{j=0}^{p-i-1} G_{i,j}(||x||^2)\mathrm{Harm}_{s-i-2j}(\mathbb{R}^m)\right) \tag{3.29}$$

and

$$\sum_{u\in X} a_u F_u(x) = g(x) + f(x)\prod_{i=1}^{p}(||x||^2 - {r_i}^2) \tag{3.30}$$

for any $x \in \mathbb{R}^m$ with some real numbers a_u, $u \in X$ and a polynomial $f(x)$ whose leading term is of degree $2(s-p)$. Then we can use Lemma 3.4 and we have

$$\sum_{u\in X} a_u G_{i,j}(||u||^2)\langle x,u\rangle^{s-i-2j} = \sum_{u\in X} a_u \left(\text{terms with } ||x||^{2l}\langle x,u\rangle^{s-i-2j-2l},\ l\geq 1\right)$$

for any i and j with $1\leq i\leq p-1$, $0\leq j\leq p-i-1$. Then, a similar argument as given in the proof of Lemma 2.4 implies

$$\sum_{u\in X} a_u G_{i,j}(||u||^2)\varphi(u) = 0,$$

for any $\varphi(x)\in \mathrm{Harm}_{s-i-2j}(\mathbb{R}^m)$. Hence by (3.29) we have

$$\sum_{u\in X} a_u g(u) = 0. \tag{3.31}$$

On the other hand, (3.30) implies $g(u) = a_u F(u) = a_u(-1)^s\prod_{i=1}^s \alpha_i^2$. Then (3.31) implies

$$(-1)^s\prod_{i=1}^s \alpha_i^2 \sum_{u\in X} a_u^2 = 0.$$

Since $(-1)^s\prod_{i=1}^s \alpha_i^2$ is a nonzero real number and $a_u^2\geq 0$, $u\in X$, we have $a_u=0$ for any $u\in X$. This completes the proof of Lemma 3.2, (i). Lemma 3.2, (ii) is obvious because the following hold:

$$\sum_{i=1}^{p-1}\sum_{j=0}^{p-i-1} G_{i,j}(||x||^2)\mathrm{Harm}_{s-i-2j}(\mathbb{R}^m) \subset \bigoplus_{i=0}^{2p-1}\mathrm{Hom}_{s-1-i}(\mathbb{R}^m)$$

and

$$\sum_{i=1}^{p-1}\sum_{j=0}^{p-1-i} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(\mathbb{R}^m) \subset \bigoplus_{i=0}^{2p-1}\mathrm{Hom}_{s+p-1-i}(\mathbb{R}^m). \qquad \blacksquare$$

Antipodal case (Proof of Theorem 1.1, (2))

A set X in $\mathbb{R}^m$ is called antipodal if $-x \in X$ holds for any $x \in X$. Let X be an antipodal s-distance set on p-concentric spheres in $\mathbb{R}^m$. Let Y be a set of all the representatives of antipodal pairs in X, i.e., Y is a subset of X satisfying $Y \cup (-Y) = X$. Thus we have $|X| = 2|Y|$. For each $u \in Y$ we define polynomials $F_u^{(e)}$ and $F_u^{(o)}$ in the following way:

$$\begin{aligned} F_u^{(e)} &= F_u + F_{-u}, \\ F_u^{(o)} &= F_u - F_{-u}, \end{aligned}$$

where $F_u(x) = \prod_{i=1}^{s}(||x-u||^2 - {\alpha_i}^2)$ which is given in page 538. We use similar notations as before. The monomials x^λ which are contained in $F_u^{(e)}$ are of even degree and the monomials x^λ which are contained in $F_u^{(o)}$ are of odd degree. We define the following sets of polynomials.

$$\begin{aligned} \mathcal{F}_Y^{(e)} &= \{F_u^{(e)} \mid u \in Y\} \\ \mathcal{F}_Y^{(o)} &= \{F_u^{(o)} \mid u \in Y\} \\ \mathcal{F}_Y^{(e)}(S) &= \{F_u^{(e)}|_S \mid u \in Y\} \\ \mathcal{F}_Y^{(o)}(S) &= \{F_u^{(o)}|_S \mid u \in Y\}. \end{aligned}$$

Then we have

$$\begin{aligned} |Y| &= \dim\left(\langle \mathcal{F}_Y^{(e)}(S)\rangle\right) = \dim\left(\langle \mathcal{F}_Y^{(o)}(S)\rangle\right), \\ \langle \mathcal{F}_X(S)\rangle &= \langle \mathcal{F}_Y^{(e)}(S)\rangle \oplus \langle \mathcal{F}_Y^{(o)}(S)\rangle. \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} \langle \mathcal{F}_Y^{(e)}(S)\rangle &\subset \bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 0 \pmod 2}} \mathrm{Hom}_{s-i}(S) + \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv 0 \pmod 2}} ||x||^{2i}\mathrm{Hom}_{s-i}(S), \\ \langle \mathcal{F}_Y^{(o)}(S)\rangle &\subset \bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 1 \pmod 2}} \mathrm{Hom}_{s-i}(S) + \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv 0 \pmod 2}} ||x||^{2i}\mathrm{Hom}_{s-i}(S), \end{aligned}$$

and

$$\begin{aligned} &\bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv \varepsilon \pmod 2}} \mathrm{Hom}_{s-i}(S) + \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv \varepsilon \pmod 2}} ||x||^{2i}\mathrm{Hom}_{s-i}(S) \\ &= \bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv \varepsilon \pmod 2}} \mathrm{Hom}_{s-i}(S) + \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv \varepsilon \pmod 2}} \sum_{j=0}^{p-1-i} ||x||^{2(i+j)}\mathrm{Harm}_{s-i-2j}(S) \end{aligned}$$

for $\varepsilon = 0, 1$.

In Lemma 3.4, we obtained polynomials $G_{i,j}(t)$, $1 \le i \le p-1$, $0 \le j \le p-i-1$ in one variable t of degree j. By definition, those polynomials depend only on $\alpha_1, \ldots, \alpha_s$, $r_1, \ldots, r_p$. We define subspaces $W^{(e)}$ and $W^{(o)}$ in $\sum_{i=1}^{p-1} ||x||^{2i} \mathrm{Hom}_{s-i}(S)$ by

$$W^{(e)} = \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv 0 (\mathrm{mod}\ 2)}}^{p-1} \sum_{j=0}^{p-i-1} G_{i,j}(||x||^2) \mathrm{Harm}_{s-i-2j}(S),$$

$$W^{(o)} = \sum_{\substack{1 \le i \le p-1 \\ s-i \equiv 1 (\mathrm{mod}\ 2)}}^{p-1} \sum_{j=0}^{p-i-1} G_{i,j}(||x||^2) \mathrm{Harm}_{s-i-2j}(S).$$

Then we have the following inequality:

$$\dim(\mathcal{F}_Y^{(e)}(S)) = \dim(\mathcal{F}_Y^{(o)}(S)) \le$$
$$\min\left\{ \dim\left(\bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 0 (\mathrm{mod}\ 2)}} \mathrm{Hom}_{s-i}(S) \right), \dim\left(\bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 1 (\mathrm{mod}\ 2)}} \mathrm{Hom}_{s-i}(S) \right) \right\}.$$

Then

$$\dim\left(\bigoplus_{\substack{0 \le i \le 2p-1 \\ s-i \equiv \varepsilon (\mathrm{mod}\ 2)}} \mathrm{Hom}_{s-i}(S) \right) = \sum_{\substack{0 \le i \le 2p-1 \\ s-i \equiv \varepsilon (\mathrm{mod}\ 2)}} \binom{m+s-i-1}{s-i}$$

for $\varepsilon = 0, 1$ and

$$\sum_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 0 (\mathrm{mod}\ 2)}} \binom{m+s-i-1}{s-i} = \begin{cases} \sum_{i=0}^{p-1} \binom{m+s-2i-1}{m-1} & \text{if } s \text{ is even,} \\ \sum_{i=0}^{p-1} \binom{m+s-2i-2}{m-1} & \text{if } s \text{ is odd,} \end{cases}$$

$$\sum_{\substack{0 \le i \le 2p-1 \\ s-i \equiv 1 (\mathrm{mod}\ 2)}} \binom{m+s-i-1}{s-i} = \begin{cases} \sum_{i=0}^{p-1} \binom{m+s-2i-2}{m-1} & \text{if } s \text{ is even,} \\ \sum_{i=0}^{p-1} \binom{m+s-2i-1}{m-1} & \text{if } s \text{ is odd.} \end{cases}$$

Since $\sum_{i=0}^{p-1} \binom{m+s-2i-2}{m-1} < \sum_{i=0}^{p-1} \binom{m+s-2i-1}{m-1}$ we have Theorem 1.1, (2). ∎

4. Examples

We give two examples of 2-distance sets whose cardinalities attain the upper bounds given in Theorem 1.1.

Example 4.1. (P. Lisoněk) Let $\mathbf{e}_i$, $1 \leq i \leq 9$ be the canonical orthonormal base of $\mathbb{R}^9$. Define subsets $X_1, X_2 \subset \mathbb{R}^9$ by

$$X_1 = \left\{ -\mathbf{e}_i + \frac{1}{3}\sum_{k=1}^{9} \mathbf{e}_k \;\middle|\; 1 \leq i \leq 9 \right\}$$
$$X_2 = \{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i < j \leq 9\}$$

Let $X = X_1 \cup X_2$. Then X is a 2-distance set whose cardinality is $\binom{8+2}{2} = 45$.

Proof. Let $H \subset \mathbb{R}^9$ be the hyper plane defined by $x_1 + x_2 + \cdots + x_9 = 2$. Then $X_1 \subset H \cap S^8_{\frac{2}{\sqrt{3}}}$ and $X_2 \subset H \cap S^8_{\sqrt{2}}$, where $S^8_\rho = \{x \in \mathbb{R}^9 \mid ||x|| = \rho\}$, that is, a sphere in $\mathbb{R}^9$ of radius ρ. It is easy to see that X is a 2-distance set on two concentric spheres in $\mathbb{R}^8$ of radius $\frac{2\sqrt{2}}{3}$ and $\frac{\sqrt{14}}{3}$. ∎

Example 4.2. Let $X = \{A = (1,\ 0),\ B = (-1,\ 0),\ C = (0,\ \sqrt{3}),\ D = (0,\ -\sqrt{3})\}$. Then X is an antipodal 2-distance set on two concentric spheres in $\mathbb{R}^2$ with cardinality 4.

Proof. It is obvious that the set X given in Example 4.2 is an antipodal 2-distance set whose cardinality coincides with the bound given in Theorem 1.1 (2). ∎

Delsarte and Seidel ([4]) gave the definition of design for Euclidean spaces in the following manner. Let X be a finite set in $\mathbb{R}^m$. Assume $0 \notin X$. Let $S_1,\ S_2, \ldots, S_p$ in $\mathbb{R}^m$ be the concentric spheres with centers at the origin satisfying the following conditions:

(1) $X \subset S_1 \cup S_2 \cup \cdots \cup S_p$,

(2) $X \cap S_i \neq \emptyset$ for $1 \leq i \leq p$.

Let $X_i = X \cap S_i$ for $1 \leq i \leq p$. Let ω be a weight function $X \ni x \longrightarrow \omega(x) \in \mathbb{R}_{>0}$. Let $\omega(X_i) = \sum_{x \in X_i} \omega(x)$. With these notation they gave the following definition.

Definition 4.3. X is a Euclidean t-design if the following condition is satisfied:

$$\sum_{i=1}^{p} \frac{\omega(X_i)}{|S_i|} \int_{\xi \in S_i} f(\xi) d\xi = \sum_{\eta \in X} f(\eta)\omega(\eta),$$

for any polynomial $f(x) = f(x_1, x_2, \ldots, x_m)$ of degree at most t, where $|S_i|$ is the area(volume) of the sphere S_i.

For the Euclidean designs, Delsarte and Seidel gave the following lower bounds for the cardinalities.

Theorem 4.4. (see [4])
(1) *Let X be a $2s$-design in $\mathbb{R}^m$, then the following holds:*

$$|X| \geq \sum_{i=0}^{2p-1} \binom{m+s-i-1}{m-1}.$$

(2) *Let X be a $(2s-1)$-design in $\mathbb{R}^m$. Assume that X is antipodal. Then the following holds:*

$$|X| \geq 2\sum_{i=0}^{p-1} \binom{m+s-2i-1}{m-1}.$$

Definition 4.5. If the cardinality of a t-design attains the lower bound given in Theorem 4.4, then we call it a tight design.

The set X given in Example 4.1 contains 45 points therefore it is not antipodal. If we concider the same configuration as given by X on two concentric spheres with center at the origin. Then we may assume X_1 is on the sphere with center at the origin and radius $\frac{2\sqrt{2}}{3}$ and X_2 is on the sphere with center at the origin and radius $\frac{\sqrt{14}}{3}$. If we define a weight function ω by

$$\omega(x) = \begin{cases} 5 & \text{if } x \in X_1, \\ 1 & \text{if } x \in X_2, \end{cases}$$

then we can check by easy calculations that Example 4.1 is a Euclidean 3-design on 2 concentric spheres in $\mathbb{R}^8$. However we can also check that there is no weight function which make Example 4.1 a Euclidean 4-design.

On the other hand Example 4.2 is an antipodal 2-distance set on 2-concentric spheres in $\mathbb{R}^2$. Define a weight function by $\omega(A) = \omega(B) = 3$, $\omega(C) = \omega(D) = 1$. Then it is easy to check that Example 4.2 is an antipodal Euclidean tight 3-design.

Theorem 4.6. *Let X be an antipodal 2-distance set on p concentric spheres in $\mathbb{R}^m$. Assume $|X| = 2m$. Then X is similar to one of the following:*
(1) *$p = 1$ and $X = \{\pm \mathbf{e}_i \mid 1 \leq i \leq m\}$.*
(2) *$p = 2$ and X is the set defined in Example 4.2.*
(Note that any antipodal 2-distance set X on p concentric spheres in $\mathbb{R}^m$ satisfies $|X| \leq 2m$.)

Proof. If $p=1$, then an antipodal 2-distance set with cardinality $2m$ is a spherical tight 3-design. It is well known that any spherical tight 3-design on $S^{m-1}(\subset \mathbb{R}^m)$ is isometric to $\{\pm\mathbf{e}_i \mid 1\le i\le m\}$. Next, assume that $p\ge 2$. We may assume that the smallest sphere among the p concentric spheres is S^{m-1}. We may also assume that $\pm\mathbf{e}_1\in X$. Let $\mathbf{a}\in X$, and $||\mathbf{a}||=r>1$. Then $A(X)$ has to be $\{2,\ 2r\}$. This implies $p=2$. Let $\mathbf{a}=(a_1,\ldots,a_m)$ and $a_1\ge 0$. If $a_1>0$, then $(1-a_1)^2+a_2^2+\cdots+a_m^2=4$ and $(1+a_1)^2+a_2^2+\cdots+a_m^2=4r^2$. This is impossible. Hence $a_1=0$. We may asume $a_2\ge 0$. Then $||\mathbf{e}_1-\mathbf{a}||=\sqrt{1+r^2}$. Since $r>1$, we have $r=\sqrt{3}$. We may assume $\mathbf{a}=(0,\sqrt{3},0,\ldots,0)$. Let $\mathbf{b}=(b_1,\ldots,b_m)\in X$ and $\mathbf{b}\ne\pm\mathbf{a},\pm\mathbf{e}_1$. We may assume $b_1\ge 0$. Then we can easily show that $b_1=0$ and $b_2=0$. If $m=2$, then X is the set defined in Example 4.2. If $m\ge 3$, then we may assume $\mathbf{b}=(0,0,b_3,0\ldots,0)$. This contradicts the assumption $|A(X)|=2$. Hence X contains at most 4 points. This completes the proof. ∎

The definiton of a Euclidean design given by Delsarte and Seidel does not give a good theory between s-distance sets and designs. Is there better definition for a Euclidean design?

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